

ON WEAKLY SEQUENTIALLY COMPLETE BANACH SPACES.

EWA M. BEDNARCZUK^{1,2}, KRZYSZTOF LEŚNIEWSKI² *

Abstract. We provide sufficient conditions for a Banach space Y to be weakly sequentially complete. These conditions are expressed in terms of the existence of directional derivatives for cone convex mappings with values in Y .

Key words. weakly sequentially complete Banach spaces, directional derivatives of cone convex mappings, cone convex mappings, directional derivatives

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1. Introduction. Weakly sequentially complete Banach spaces were introduced in [2]. Since then, properties of weakly sequentially complete spaces were investigated in a number of papers, see e.g. [5] and [22] and the references therein. Important result on weakly sequentially complete Banach spaces is given by Rosenthal in [22].

THEOREM 1.1. *If Y is weakly sequentially complete, then Y is either reflexive or contains a subspace isomorphic to l^1 .*

In the present paper we prove sufficient conditions for a Banach space Y to be weakly sequentially complete. These sufficient conditions are expressed in terms of existence of directional derivatives for cone convex mappings.

Cone convex mappings appear in variational analysis in topological vector spaces and in construction of efficient iterative schemes for solving vector optimization problems. Newton method for solving smooth unconstrained vector optimization problems under partial order induced by general closed convex pointed cone can be found in [8].

The organization of the paper is as follows. In Section 2 we discuss basic facts concerning weakly sequentially complete Banach spaces. In Section 3 we prove some properties of cone convex mappings. Section 4 is devoted to the constructions of convex functions that take values in given points. Section 5 contains the main result namely the proof of the fact that the existence of directional derivatives at any point and any direction for any cone convex mapping implies that the Banach space Y is weakly sequentially complete. In Section 6 we discuss properties of the mapping F constructed in Theorem 5.2.

2. Preliminary facts. Let us present some known facts about weakly sequentially complete Banach spaces.

DEFINITION 2.1. *A sequence $\{y_n\}$ in a Banach space Y is weak Cauchy if $\lim_{n \rightarrow \infty} y^*(y_n)$ exists for every $y^* \in Y^*$. We say that a Banach space Y is weakly sequentially complete if every weak Cauchy sequence weakly converges in Y . A weak Cauchy sequence $\{y_n\}$ is called nontrivial if it does not weakly converge.*

Nontrivial weak Cauchy sequences were investigated by [9, 14, 19]. Any weak Cauchy sequence $\{y_n\}$ in a Banach space Y is norm-bounded by the Uniform Boundedness principle ([1], p. 38). In [6] we can find a well known fact about reflexive space ([6] Corollary 4.4).

PROPOSITION 2.2 ([6]). *If Y is reflexive, then Y is weakly sequentially complete.*

*1 Systems Research Institute, Polish Academy of Sciences, ul. Newelska 6, 01-447 Warsaw, Poland, 2 Warsaw University of Technology, Faculty of Mathematics and Information Science, ul. Koszykowa 75, 00-662 Warsaw, Poland, (email addresses: e.bednarczuk@mini.pw.edu.pl, k.lesniewski@mini.pw.edu.pl).

However, in some nonreflexive spaces, sequences can be found which are weak Cauchy but not weakly convergent. Let us present some examples, c.f. [6, 17].

EXAMPLE 1.

- Let $Y = c_0$ and let us consider $y_n := \sum_{k=1}^n e_k$, where e_k is basic vector in c_0 . We have $(c_0)^* = l^1$ and $\{y_n\}$ is weak Cauchy sequence. Since $\{y_n\}$ converges weak* to the element $(1, 1, \dots)$, it is not weakly convergent.
- If we take K which is a compact Hausdorff space, then $C(K)$ is weakly sequentially complete if and only if K is finite.
- Space $C[0, 1]$ of all continuous functions on $[0, 1]$ is not weakly sequentially complete.
- For every measure μ , the space $L^1(\mu)$ is weakly sequentially complete.

PROPOSITION 2.3. Any closed subspace $Z \subset Y$ of a weakly sequentially complete Banach space Y is weakly sequentially complete Banach space.

Proof. Consider a weak Cauchy sequence $\{y_n\}$ in Z . Then it is weak Cauchy in Y . Since Y is weakly sequentially complete, $\{y_n\}$ weakly converges to some $y \in Y$. Thus, y lies in the weak closure of Z . Since Z is closed and convex, by Mazur's theorem, weak closure of Z coincides with Z . Hence $y \in Z$. \square

PROPOSITION 2.4. Let $\{y_n\} \in Y$ be a nontrivial weak Cauchy sequence. Then each subsequence of $\{y_n\}$ is also nontrivial weak Cauchy.

Proof. By contradiction, assume that there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ weakly converging to $y_0 \in Y$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbf{N}$ such that for all $k > N_1$ $|f^*(y_{n_k}) - f^*(y_0)| \leq \frac{\varepsilon}{2}$. Since $\{y_n\}$ is weak Cauchy, there exists $N \in \mathbf{N}$ such that $|f^*(y_n) - f^*(y_{n_k})| \leq \frac{\varepsilon}{2}$ for all $n > N$. Since $k, n > \max\{N_1, N\}$ we have

$$\begin{aligned} |f^*(y_n) - f^*(y_0)| &= |f^*(y_n) - f^*(y_{n_k}) + f^*(y_{n_k}) - f^*(y_0)| \leq \\ &|f^*(y_n) - f^*(y_{n_k})| + |f^*(y_{n_k}) - f^*(y_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

\square

In the present paper we consider cone ordering relations in Y generated by convex closed cones. If we assume that Y is a Banach lattice, then we get the following characterization ([18]).

THEOREM 2.5 ([18]). If Y is a Banach lattice, then Y is weakly sequentially complete if and only if it does not contain any subspace isomorphic to c_0 .

Let us underline that if we consider the concept of weak completeness in terms of nets (see [10] for the definition), then we cannot expect weak completeness in the infinite dimensional case. A net $S := \{x_\sigma, \sigma \in \Sigma\}$ in a topological vector space Y is Cauchy if for every neighbourhood U of zero there exists $\sigma_0 \in \Sigma$ such that $x_{\sigma_1} - x_{\sigma_2} \in U$ for every $\sigma_1, \sigma_2 \geq \sigma_0$. Topological vector space Y is complete if each Cauchy net converges (see [13] p.356).

First countable spaces, including metric spaces, have topologies that are determined by their convergent sequences. In an arbitrary first countable space, a point x is in the closure of a subset A if and only if there is a sequence in A converging to x .

If Y is an infinite dimensional Banach space, regardless of whether or not it is separable in the norm topology, then the weak topology on Y is not first countable, and is not characterized by its convergent sequences alone.

For the weak topology in an infinite-dimensional space, there exist weak Cauchy nets which are not weakly convergent. This result can be found in [7, 9] and is deeply tied with Axiom of choice and Helly's Theorem.

PROPOSITION 2.6 ([9] p.14). The weak topology on infinite-dimensional normed space is never complete.

3. Cone-convex mappings. Let X, Y be real linear vector spaces and let $K \subset Y$ be a convex cone inducing the standard ordering relation

$$x \leq_K y \Leftrightarrow y - x \in K \quad \text{for any } x, y \in Y.$$

Analogously, we write $y \geq_K x$ if and only if $y - x \in K$. We use the notation \geq if K is clear from the context.

Let $F : X \rightarrow Y$. We say that the mapping F is K -convex on a convex set $A \subset X$ if

$$(3.1) \quad F(\lambda x + (1 - \lambda)y) \leq_K \lambda F(x) + (1 - \lambda)F(y) \quad \text{for all } x, y \in A \text{ and } \lambda \in [0, 1].$$

As in the scalar case [24] we have the following characterization of cone convexity of F in terms of the epigraph of F , $\text{epi}F$, defined as $\text{epi}F := \{(x, y) \in A \times Y : y \geq_K F(x)\}$.

PROPOSITION 3.1. *Let $K \subset Y$ be a closed convex cone. A mapping $F : X \rightarrow Y$ is K -convex on $A \subset X$ if and only if $\text{epi}F$ is a convex set in $A \times Y$.*

Proof. Let us assume that F is K -convex on A . Let $(x_1, y_1), (x_2, y_2) \in \text{epi}F$ and $\lambda \in [0, 1]$. From the definition of $\text{epi}F$ we have $y_1 - F(x_1) = k_1$ and $y_2 - F(x_2) = k_2$ for some $k_1, k_2 \in K$. Since K is convex, $\lambda k_1 + (1 - \lambda)k_2 \in K$. From K -convexity of F on A we get

$$F(\lambda x_1 + (1 - \lambda)x_2) \leq_K \lambda F(x_1) + (1 - \lambda)F(x_2) \leq_K \lambda y_1 + (1 - \lambda)y_2.$$

The converse implication follows directly from the definition of $\text{epi}F$. \square

Let Y be a Banach space with the dual space Y^* . Let K be a closed convex cone in Y . Dual cone K^* of K is defined as $K^* := \{y^* \in Y^* : y^*(y) \geq 0 \forall y \in K\}$.

PROPOSITION 3.2 ([12], Lemma 3.21). *Let $K \subset Y$ be a closed convex cone in Y . The cone K can be represented as*

$$(3.2) \quad K = \{y \in Y : y^*(y) \geq 0 \forall y^* \in K^*\}.$$

The following lemma provides another characterization of K -convex mappings. The finite-dimensional case of this lemma has been used in [20].

LEMMA 3.3. *Let $A \subset X$ be a convex subset of X . Let $K \subset Y$ be a closed convex cone and let $F : X \rightarrow Y$ be a mapping. The following conditions are equivalent.*

1. *The mapping F is K -convex on A .*
2. *For any $u^* \in K^*$, the composite function $u^*(F) : A \rightarrow \mathbf{R}$ is convex.*

Proof. The implication \Rightarrow follows directly from the definition of K^* .

We prove the converse implication by contradiction. Suppose that for some $\lambda \in [0, 1]$, $x, y \in A$ we have $\lambda F(x) + (1 - \lambda)F(y) - F(\lambda x + (1 - \lambda)y) \notin K$. By (3.2), there exists $u^* \in K^*$ such that $u^*(\lambda F(x) + (1 - \lambda)F(y) - F(\lambda x + (1 - \lambda)y)) < 0$ which, by Proposition 3.2, contradicts the convexity of $u^*(F)$ on A . \square

4. Construction of Convex Functions. In this section we construct convex function which take given values at a countable number of points. Some constructions of convex functions are present in the literature. For example, in [16] we can find the proof that a continuous convex function f defined on l_p , $p \geq 1$,

$$f(x) := \|(|x_1|^{1+1}, |x_2|^{1+1/2}, \dots)\|_p,$$

is everywhere compactly differentiable but not Fréchet differentiable at zero. In [3] we can find an extension of a function $\Phi : X \rightarrow \bar{\mathbf{R}}$ to some convex function $\eta : C \rightarrow \bar{\mathbf{R}}$, $X \subset C$, C is a convex set, such that $\eta(x) = \Phi(x)$ for all $x \in X$.

Let us start with the following proposition.

PROPOSITION 4.1. *For every nonnegative decreasing and convergent sequence $\{a_m\} \subset \mathbf{R}$ there exists a subsequence $\{a_{m_k}\} \subset \{a_m\}$, $k \in \mathbf{N}$, and a convex nonincreasing function $g : \mathbf{R} \rightarrow \mathbf{R}$ such that $g(m_k) = a_{m_k}$.*

Proof. Let $a_{m_1} := a_1$, $a_{m_2} := a_2$. Let $f_1 : \mathbf{R} \rightarrow \mathbf{R}$ be defined as

$$f_1(x) := a_{m_1} + \frac{x-1}{m_2-1}(a_{m_2} - a_{m_1}).$$

Let x_2 be such that $f_1(x_2) = 0$. We have $x_2 > m_2$. By taking $m_3 := [x_2] + 1$ where $[x]$ is the largest integer not greater than x . We define $f_2 : \mathbf{R} \rightarrow \mathbf{R}$ as

$$f_2(x) := a_{m_2} + \frac{x-m_2}{m_3-m_2}(a_{m_3} - a_{m_2}).$$

Suppose that we have already defined f_1, \dots, f_k , $k \geq 2$. We define f_{k+1} in the following way. Let x_k be such that $f_k(x_k) = 0$. We have $x_k > m_k$. By taking $m_{k+2} = [x_k] + 1$ we define $f_{k+1} : \mathbf{R} \rightarrow \mathbf{R}$ as

$$f_{k+1}(x) := a_{m_{k+1}} + \frac{x-m_{k+1}}{m_{k+2}-m_{k+1}}(a_{m_{k+2}} - a_{m_{k+1}}).$$

We define $g : \mathbf{R} \rightarrow \mathbf{R}$ by formula $g(x) = \sup_k f_k(x)$. Clearly function g is convex on \mathbf{R} . Let us show that

$$(4.1) \quad \begin{aligned} f_{k+1}(x) &\geq f_k(x) && \text{for } x \geq m_{k+1} \\ f_{k+1}(x) &\leq f_k(x) && \text{for } x \leq m_{k+1}. \end{aligned}$$

By construction of m_{k+1} we have $a_{m_{k+2}} = f_k(m_{k+2})$ and

$$a_{m_{k+2}} = a_{m_k} + \frac{m_{k+2}-m_k}{m_{k+1}-m_k}(a_{m_{k+1}} - a_{m_k}) \geq a_{m_k} + \frac{m_{k+2}-m_{k+1}}{m_{k+1}-m_k}(a_{m_{k+1}} - a_{m_k})$$

next from monotonicity of $\{a_m\}$ we get

$$\frac{a_{m_{k+1}} - a_{m_{k+2}}}{m_{k+2} - m_k} \leq \frac{a_{m_k} - a_{m_{k+2}}}{m_{k+2} - m_k} \leq \frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k}$$

and

$$(4.2) \quad \frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k} \geq \frac{a_{m_{k+1}} - a_{m_{k+2}}}{m_{k+2} - m_{k+1}}.$$

To get (4.1) we multiply above inequality by $m_{k+1} - x$. Let us assume that $m_{k+1} \geq x$ then from (4.2)

$$\begin{aligned} \frac{a_{m_k} - a_{m_{k+1}}}{m_{k+1} - m_k}(m_{k+1} - x) + \frac{a_{m_{k+2}} - a_{m_{k+1}}}{m_{k+2} - m_{k+1}}(m_{k+1} - x) &\geq 0 \\ (a_{m_k} - a_{m_{k+1}}) \left(1 - \frac{x-m_k}{m_{k+1}-m_k}\right) + \frac{a_{m_{k+2}} - a_{m_{k+1}}}{m_{k+2}-m_{k+1}}(m_{k+1} - x) &\geq 0 \end{aligned}$$

we get $f_k(x) \geq f_{k+1}(x)$. Analogous we can prove first inequality in (4.1) for $x \geq m_{k+1}$.

By (4.1), function g is decreasing and $g(m_k) = a_{m_k}$. \square

In Proposition 4.1 we constructed function g when $\{a_m\}$ is nonnegative, decreasing and $\{t_k\}$, $t_k = m_k$, $k \in \mathbf{N}$ is a subsequence of integers. Now we consider the case when $\{a_m\}$ is arbitrary and $\{t_k\}$ is a decreasing sequence of reals.

LEMMA 4.2. *Let $\{a_m\}, \{t_m\} \subset \mathbf{R}$ be sequences with $\{t_m\}$ decreasing. If*

$$(4.3) \quad \frac{a_{m+1} - a_m}{t_{m+1} - t_m} \geq \frac{a_{m+2} - a_{m+1}}{t_{m+2} - t_{m+1}} \quad \text{for } m \in \mathbf{N},$$

there exists a convex function $g : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$g(t_m) = a_m.$$

Proof. For each $m \in \mathbf{N}$, let $f_m : \mathbf{R} \rightarrow \mathbf{R}$ be defined as

$$f_m(x) := a_m + \frac{x - t_m}{t_{m+1} - t_m}(a_{m+1} - a_m) \quad \text{for } x \in \mathbf{R}.$$

We show that for $m \in \mathbf{N}$ we have

$$(4.4) \quad \begin{aligned} f_{m+1}(x) &\leq f_m(x) & \text{for } x \geq t_{m+1} \\ f_{m+1}(x) &\geq f_m(x) & \text{for } x \leq t_{m+1}. \end{aligned}$$

If $x \geq t_{m+1}$ then multiplying both sides of (4.3) by $t_{m+1} - x$ we get

$$(a_{m+1} - a_m) \left(1 - \frac{x - t_m}{t_{m+1} - t_m} \right) + \frac{x - t_{m+1}}{t_{m+2} - t_{m+1}}(a_{m+2} - a_{m+1}) \leq 0$$

which is equivalent to $f_{m+1}(x) \leq f_m(x)$. Analogously if $x \leq t_{m+1}$ we get $f_{m+1}(x) \geq f_m(x)$.

We show that the function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined as

$$g(r) = \sup_m f_m(r)$$

satisfies the requirements of the Lemma 4.2 on \mathbf{R} . Indeed, g is convex as supremum of convex functions. By the first inequality of (4.4) and the monotonicity of $\{t_m\}$ we get

$$f_{m+k}(t_m) \leq f_m(t_m) = a_m \quad \text{for } k \geq 1.$$

By the second inequality of (4.4) we get

$$f_{m-k}(t_m) \leq f_m(t_m) \quad \text{for } k = 1, \dots, m-1.$$

This gives that $g(t_m) = a_m$. \square

Observe that, we have the following property:

$$(4.5) \quad g(x) = f_m(x) \quad \text{for } t_{m+1} \leq x \leq t_m.$$

Indeed, by the first inequality of (4.4) and the monotonicity of $\{t_m\}$ we get

$$f_m(x) \geq f_{m-1}(x) \geq f_{m-2}(x) \geq \dots f_1(x).$$

By the second inequality of (4.4) and from the fact that $x \geq t_{m+k}$, $k \geq 1$ we get

$$f_{m+k}(x) \leq f_m(x) \quad \text{for } k \geq 1.$$

We apply the above lemma to construct a convex function g starting with any nonnegative converging sequence $\{z_m\}$ with positive limit.

PROPOSITION 4.3. *Let $\{z_m\} \subset \mathbf{R}$ be a converging sequence of nonnegative reals with limit $z > 0$.*

There exist a subsequence $\{z_{m_k}\} \subset \{z_m\}$, a sequence $\{t_k\} \subset \mathbf{R}$, $t_k \downarrow 0$ and a convex function $g : \mathbf{R} \rightarrow \mathbf{R}$, such that

$$g(t_k) = a_k,$$

where

$$\begin{aligned} a_k &:= t_k z_{m_k} \text{ if } \{z_{m_k}\} \text{ is nonincreasing and } 0 < z < 1, \\ a_k &:= t_k z_{m_k} \frac{1}{q}, \quad q > z, \text{ if } \{z_{m_k}\} \text{ is nonincreasing and } z > 1, \\ a_k &:= -t_k z_{m_k}, \text{ if } \{z_{m_k}\} \text{ is increasing and } z > 1, \\ a_k &:= -t_k z_{m_k} \frac{1}{q}, \quad q < z, \text{ if } \{z_{m_k}\} \text{ is increasing and } 0 < z < 1. \end{aligned}$$

Proof. We can assume that $z_m > 0$ for $m \in \mathbf{N}$. Two cases should be considered:

1. $\{z_m\}$ contains an infinite subsequence $\{z_{m_k}\} \subset \{z_m\}$,

$$\{z_{m_k}\} \text{ is nonincreasing, i.e. } z_{m_{k+1}} \leq z_{m_k}, \quad k \in \mathbf{N},$$

2. $\{z_m\}$ contains an infinite subsequence $\{z_{m_k}\} \subset \{z_m\}$,

$$\{z_{m_k}\} \text{ is increasing, i.e. } z_{m_{k+1}} > z_{m_k}, \quad k \in \mathbf{N}.$$

To prove the assertion we show that in both cases condition (4.3) of Lemma 4.2 is satisfied.

Case 1. Without loss of generality we can assume that $z_{m_k} = z_m$. Moreover, we can assume that $\{z_m\}$ is decreasing (in case when $\{z_m\}$ is constant the existence of the function g satisfying the requirements of the proposition follows trivially from Lemma 4.2).

Case 1a. Consider first the case when $0 < z < 1$. By eliminating eventually a finite number of z_m we can assume that $0 < z_m < 1$ for $m \in \mathbf{N}$.

Let $t_m := (z_m)^m$ and $a_m := (z_m)^{m+1}$ for $m \in \mathbf{N}$. Both sequences are decreasing. Let $m_1 := 1$, $m_2 := 2$. Since $\{z_m\}$ and $\{a_m\}$ are decreasing, the number C defined below is positive, i.e.

$$C := \frac{t_1 t_2}{a_2 - a_1} (z_2 - z_1) > 0.$$

Since $t_m \rightarrow 0$, there exists $K \geq m_2$ such that

$$t_m < C \quad \text{for } m > K.$$

Put $m_3 := K + 1$. We have

$$\begin{aligned} t_{m_3} &< \frac{t_1 t_2}{a_2 - a_1} (z_2 - z_1) = \frac{t_1 a_2 - t_2 a_1 + t_2 a_2 - t_2 a_2}{a_2 - a_1} = \frac{t_2 (a_2 - a_1) - a_2 (t_2 - t_1)}{a_2 - a_1} \\ &= t_2 - \frac{a_2 (t_2 - t_1)}{a_2 - a_1} < t_2 - \frac{a_2 (t_2 - t_1)}{a_2 - a_1} + \frac{a_{m_3} (t_2 - t_1)}{a_2 - a_1}. \end{aligned}$$

The last inequality follows from the fact that $\frac{a_{m_3} (t_2 - t_1)}{a_2 - a_1} > 0$. Thus,

$$t_{m_3} - t_2 < \frac{(a_{m_3} - a_2)(t_2 - t_1)}{a_2 - a_1}$$

which proves that

$$\frac{a_2 - a_1}{t_2 - t_1} \geq \frac{a_{m_3} - a_2}{t_{m_3} - t_2}.$$

Assume that we have already defined m_1, m_2, \dots, m_k , $k \geq 3$. By induction with respect to k , starting from m_{k-1} and m_k we choose m_{k+1} by repeating the reasoning above with m_{k-1} and m_k instead m_1 and m_2 , respectively, and with $K \geq m_k$. In this way we choose m_{k+1} and we prove condition (4.3) of Lemma 4.2.

Case 1b. When $z > 1$ it is enough to take $t_m := \left(\frac{zm}{q}\right)^m$ and $a_m := t_m z_m \frac{1}{q}$ for arbitrary $q > z$ for $m \in \mathbf{N}$. Again both sequences are decreasing. Hence, we can repeat the above reasoning with these sequences and get the existence of the function g satisfying the requirements.

Case 2. As previously, without loss of generality, we can assume that $z_{m_k} = z_m$.

Case 2a. Consider the case $z > 1$. By eliminating eventually a finite number of z_m we can assume that $z_m > 1$ for $m \in \mathbf{N}$.

Let $t_m := \frac{1}{(z_m)^m}$ and $a_m := -t_m z_m = -\frac{1}{(z_m)^{m-1}}$.

Let $m_1 := 1$, $m_2 := 2$. Since $\{t_m\}$ is decreasing and $\{z_m\}$ is increasing, the number C defined below is negative, i.e.

$$C := \frac{t_1 t_2}{t_2 - t_1} (z_2 - z_1) < 0.$$

Since $a_m \rightarrow 0$ there exists $K \geq m_2$ such that

$$a_m > C \quad \text{for } m > K.$$

Put $m_3 := K + 1$. We have

$$\begin{aligned} a_{m_3} &\geq \frac{t_1 t_2}{t_2 - t_1} (z_2 - z_1) = \frac{t_2 a_1 - t_1 a_2 + t_2 a_2 - t_1 a_2}{t_2 - t_1} = \frac{a_2(t_2 - t_1) - t_2(a_2 - a_1)}{t_2 - t_1} \\ &= a_2 - \frac{t_2(a_2 - a_1)}{t_2 - t_1} > a_2 - \frac{t_2(a_2 - a_1)}{t_2 - t_1} + \frac{t_{m_3}(a_2 - a_1)}{t_2 - t_1}. \end{aligned}$$

The last inequality follows from the fact that $\frac{t_{m_3}(a_2 - a_1)}{t_2 - t_1} < 0$. Thus,

$$a_{m_3} - a_2 > \frac{(t_{m_3} - t_2)(a_2 - a_1)}{t_2 - t_1}.$$

As previously, this proves that

$$\frac{a_2 - a_1}{t_2 - t_1} \geq \frac{a_{m_3} - a_2}{t_{m_3} - t_2}.$$

Assume that we have already defined m_1, m_2, \dots, m_k , $k \geq 3$. By induction with respect to k , starting from m_{k-1} and m_k we choose m_{k+1} by repeating the reasoning above with m_{k-1} and m_k instead m_1 and m_2 , respectively, and with $K \geq m_k$. In this way we choose m_{k+1} and we prove condition (4.3) of Lemma 4.2.

Case 2b. When $0 < z < 1$ it is enough to take $t_m := \frac{1}{\left(\frac{zm}{q}\right)^m}$ for some $q < z$ and $a_m := -t_m z_m \frac{1}{q}$. We can repeat the above reasoning with these sequences and get the existence of the function g satisfying the requirements.

In consequence, in all cases, by Lemma 4.2, we get the function $g : \mathbf{R} \rightarrow \mathbf{R}$ defined by formula $g(r) = \sup_k f_k(r)$ such that $g(t_k) = a_k$, where f_k , $k \in \mathbf{N}$, are as in the proof of Lemma 4.2. \square

Proposition 4.3 allows us to formulate the following corollary used in the proof of the main result (Theorem 5.2).

PROPOSITION 4.4. *Let Y be a normed space and let $\{y_m\} \subset Y$ be such that the sequence $\{\bar{y}^*(y_m)\}$ converges to $z \in \mathbf{R} \setminus \{0\}$ for a certain $\bar{y}^* \in Y^* \setminus \{0\}$.*

There exist a subsequence $\{y_{m_k}\} \subset \{y_m\}$, a functional $y^ \in Y^*$ and a convex function $g : \mathbf{R} \rightarrow \mathbf{R}$, such that*

$$g(t_k) = a_k,$$

where

1. $t_k := (y^*(y_{m_k}))^{m_k}$, $a_k := t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $0 < z < 1$,
2. $t_k := (\frac{y^*(y_{m_k})}{q})^{m_k}$, $a_k := t_k y^*(y_{m_k}) \frac{1}{q}$, for some $q > z$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $z > 1$,
3. $t_k := (\frac{1}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is increasing and $z > 1$,
4. $t_k := (\frac{q}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k}) \frac{1}{q}$ for some $q < z$, if $\{y^*(y_{m_k})\}$ is increasing and $0 < z < 1$.

Proof. The proof follows directly from Proposition 4.3. Indeed, let us observe that we can always find some $y^* \in Y^*$, $y^* = c\bar{y}^*$ for a certain $c \neq 0$ such that $\{y^*(y_m)\}$ is a converging sequence of nonnegative reals with the limit point $z > 0$. \square

5. Main results. Let X be a vector space and Y be a real Banach space. Let $F : A \rightarrow Y$ be a mapping defined on a subset A of X .

DEFINITION 5.1. *We say that the mapping $F : A \rightarrow Y$ is directionally differentiable at $x_0 \in A$ in the direction $h \neq 0$ such that $x_0 + th \in A$ for all t sufficiently small if the limit*

$$(5.1) \quad F'(x_0; h) := \lim_{t \downarrow 0} \frac{F(x_0 + th) - F(x_0)}{t}$$

exists. The element $F'(x_0; h)$ is called the directional derivative of F at x_0 in the direction h .

Let $A \subset X$ be a convex subset of X . Let $F : A \rightarrow Y$ be a K -convex mapping on A , where $K \subset Y$ is a closed convex cone in Y . By elementary calculations [23] one can

prove that for any K -convex mappings F on A , for all $x_0 \in A$, $h \in X$, $x_0 + th \in A$ for all t sufficiently small, the difference quotient $q(t) := \frac{F(x_0 + th) - F(x_0)}{t}$ is nondecreasing as a function of t .

Our aim is to prove the following theorem.

THEOREM 5.2. *Let X be a real linear vector space and let Y be a Banach space. If for every closed convex cone $K \subset Y$ and every K -convex mapping $F : A \rightarrow Y$, $A \subset X$ is a convex subset of X , $0 \in A$, the directional derivative $F'(0; h)$ exists for any $h \in X$, $h \neq 0$ such that $th \in A$ for all t sufficiently small, then Y is weakly sequentially complete.*

Proof. By contradiction, assume that Y is not weakly sequentially complete. This means that there exists a nontrivial weak Cauchy sequence $\{y_m\}$ in Y , i.e.

$$\text{for all } y^* \in Y^* \text{ the sequence } y^*(y_m) \text{ converges}$$

and $\{y_m\}$ is not weakly convergent in Y . Basing ourselves on the existence of the above sequence we construct a closed convex cone $K \subset Y$ and a K -convex mapping $F : A \rightarrow Y$ such that for a given direction $0 \neq h \in X$

$$F'(0; h) \text{ does not exist.}$$

Let us observe first that we can always find a functional $y^* \in Y^* \setminus \{0\}$ such that

$$(5.2) \quad y^*(y_m) \rightarrow z \in \mathbf{R} \setminus \{0\},$$

since otherwise $\{y_m\}$ would weakly converge to zero. Let us note that without losing generality we can assume that $z > 0$. Consequently, by neglecting eventually a finite number of elements we can also assume that $y^*(y_m) > 0$, $m \in \mathbf{N}$.

Let $K^* \subset Y^*$ be defined as

$$K^* := \cup_{\lambda \geq 0} \lambda y^*.$$

The cone K^* is a half-line emanating from 0 in the direction of y^* . This is a closed pointed convex cone in Y^* .

Let

$$(5.3) \quad K := \{y \in Y \mid z^*(y) \geq 0 \text{ for all } z^* \in K^*\}.$$

By (5.2), $\{y_m\}_{m \in \mathbf{N}} \subset K$. The cone K is closed and convex. By Proposition 4.4, there exist a sequence $t_k \downarrow 0$, a subsequence $\{y_{m_k}\} \subset \{y_m\}$ and a convex function $g : \mathbf{R} \rightarrow \mathbf{R}$ such that $g(t_k) = a_k$, where

1. $t_k := (y^*(y_{m_k}))^{m_k}$, $a_k := t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $0 < z < 1$,
2. $t_k := (\frac{y^*(y_{m_k})}{q})^{m_k}$, $a_k := t_k y^*(y_{m_k}) \frac{1}{q}$, where $q > z$, if $\{y^*(y_{m_k})\}$ is nonincreasing and $z > 1$,
3. $t_k := (\frac{1}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k})$, if $\{y^*(y_{m_k})\}$ is increasing and $z > 1$,
4. $t_k := (\frac{q}{y^*(y_{m_k})})^{m_k}$, $a_k := -t_k y^*(y_{m_k}) \frac{1}{q}$ for some $q < z$, if $\{y^*(y_{m_k})\}$ is increasing and $0 < z < 1$.

Now we construct K -convex mappings $F : A \rightarrow Y$ satisfying the requirements of the theorem, where $A := \{x \in X : x = rh, r \geq 0\}$ for some $h \in X$, $h \neq 0$. We limit our attention to the case 1. and the case 3.. The case 2. and the case 4. require only minor changes.

Case 1. We construct a K -convex mapping $F : A \rightarrow Y$ such that

$$F(t_k h) = a_k = t_k y_{m_k}, \quad k \in \mathbf{N},$$

where $\{t_k\}$ and $\{y_{m_k}\}$ are such that $\{t_k\}$ and $\{y^*(y_{m_k})\}$ are as in the case 1. Let $F_k : A \rightarrow Y$, $k \in \mathbf{N}$, be defined as follows. Let $F_1 : A \rightarrow Y$ be defined as

$$F_1(x) := \begin{cases} y_{m_1} t_1 + \frac{r-t_1}{t_2-t_1} (y_{m_2} t_2 - y_{m_1} t_1) & t_2 < r \\ 0 & r < t_2. \end{cases}$$

For any $k \in \mathbf{N}$, $k \geq 2$ and any $x = rh$, $r > 0$ we define $F_k : A \rightarrow Y$ as follows

$$F_k(x) := \begin{cases} y_{m_k} t_k + \frac{r-t_k}{t_{k+1}-t_k} (y_{m_{k+1}} t_{k+1} - y_{m_k} t_k) & t_{k+1} < r \leq t_k \\ 0 & r \notin (t_{k+1}, t_k]. \end{cases}$$

We start by showing that the mapping $F : A \rightarrow Y$ defined as $F(x) := \sum_{i=1}^{\infty} F_i(x)$ is well defined. To this aim it is enough to observe that for any $x \in A$ with $x = rh$, where $t_{k+1} < r \leq t_k$, for any $N \in \mathbf{N}$ we have

$$\sum_{i=1}^N F_i(x) = \begin{cases} F_k(x) & N \geq k \\ 0 & N < k. \end{cases}$$

Consequently, for $t_{k+1} < r \leq t_k$, we have

$$(5.4) \quad F(x) = \sum_{i=1}^{\infty} F_i(x) = \lim_{N \rightarrow \infty} \sum_{i=1}^N F_i(x) = F_k(x).$$

It is easy to see that for $r > t_2$ we have $F(x) = F_1(x)$. Moreover, $F(0) = 0$. We show that the mapping F is K -convex on A with respect to cone K defined by (5.3). To this aim we use Lemma 3.3,

Precisely, we start by showing that for $x \in A$, $x = rh$, $r > 0$ we have

$$y^*(F(x)) = g(r),$$

where function g is as in Proposition 4.4. Indeed, if $r \in (t_{k+1}, t_k]$ for some $k \geq 2$, then by (4.5) and (5.4),

$$y^*(F(x)) = y^*(F_k(rh)) = y^*\left(y_{m_k}t_k + \frac{r - t_k}{t_{k+1} - t_k}(y_{m_{k+1}}t_{k+1} - y_{m_k}t_k)\right) = f_k(r) = g(r).$$

If $r > t_2$, then

$$y^*(F(x)) = y^*(F_1(rh)) = y^*\left(y_{m_1}t_1 + \frac{r - t_1}{t_2 - t_1}(y_{m_2}t_2 - y_{m_1}t_1)\right) = f_1(r) = g(r).$$

Now, take any $z^* \in K^*$. By the definition of K^* , $z^* = \beta y^*$ for some $\beta \geq 0$ and by convexity of g we get

$$(5.5) \quad \begin{aligned} z^*(F((\lambda a_1 + (1 - \lambda)a_2)h)) &= \beta y^*(F((\lambda a_1 + (1 - \lambda)a_2)h)) \\ &= \beta g(\lambda a_1 + (1 - \lambda)a_2) \leq \beta \lambda g(a_1) + \beta(1 - \lambda)g(a_2) = \\ &= \lambda z^*(F(a_1h)) + (1 - \lambda)z^*(F(a_2h)). \end{aligned}$$

By Lemma 3.3, this proves that F is K -convex on A .

Case 2. We take $t_k := (\frac{y^*(y_{m_k})}{q})^{m_k}$ and $a_k := t_k y^*(y_{m_k})^{\frac{1}{q}}$, where $q > z$. With this choice $\{t_k\}$, $\{a_k\}$, $k \in \mathbf{N}$, by repeating the reasoning from Case 1. we get the required mapping F .

Case 3. We construct a K -convex mapping $F : A \rightarrow Y$ such that

$$F(t_k h) = a_k = -t_k y_{m_k}, \quad k \in \mathbf{N},$$

where $\{t_k\}$ and $\{y_{m_k}\}$ are such that $\{t_k\}$ and $\{y^*(y_{m_k})\}$ are as in the case 3.

Let $F_k : A \rightarrow Y$, $k \in \mathbf{N}$, be defined as follows. Let $F_1 : A \rightarrow Y$ be defined as

$$F_1(x) := \begin{cases} -y_{m_1}t_1 - \frac{r-t_1}{t_2-t_1}(y_{m_2}t_2 - y_{m_1}t_1) & t_2 < r \\ 0 & r < t_2. \end{cases}$$

For any $k \in \mathbb{N}$, $k \geq 2$ and any $x \in A$, $x = rh$, $r > 0$ we define $F_k : A \rightarrow Y$ as follows

$$F_k(x) := \begin{cases} -y_{m_k}t_k - \frac{r-t_k}{t_{k+1}-t_k}(y_{m_{k+1}}t_{k+1} - y_{m_k}t_k) & t_{k+1} < r \leq t_k \\ 0 & r \notin (t_{k+1}, t_k]. \end{cases}$$

Now the mapping $F : A \rightarrow Y$ is defined as in Case 1. by formula (5.4). As previously, for $r > t_2$ we have $F(x) = F_1(x)$ and $F(0) = 0$. For any $z^* \in K^*$, the convexity of function $z^*(F)$ is proved by the same arguments as in Case 1. in the proof of (5.5).

Case 4. We have $t_k := (\frac{q}{y^*(y_{m_k})})^{m_k}$ and $a_k := -t_k y^*(y_{m_k})^{\frac{1}{q}}$ for some $q < z$. With this choice $\{t_k\}$, $\{a_k\}$, $k \in \mathbb{N}$, by repeating the reasoning from Case 3. we get the required mapping F .

In all the cases, the directional derivative of F at zero in the direction $h \in X$ equals

$$F'(0; h) = \lim_{t \downarrow 0} \frac{F(th)}{t}.$$

Summing up the considerations above we get the following formulas:

$$\begin{aligned} \text{Case 1. } F'(0; h) &= \lim_{k \rightarrow +\infty} \frac{F(t_k h)}{t_k} = \lim_{k \rightarrow +\infty} y_{m_k}, \\ \text{Case 2. } F'(0; h) &= \lim_{k \rightarrow +\infty} \frac{F(t_k h)}{t_k} = \lim_{k \rightarrow +\infty} \frac{1}{q} y_{m_k} \quad \text{for some } q > z. \\ \text{Case 3. } F'(0; h) &= \lim_{k \rightarrow +\infty} \frac{F(t_k h)}{t_k} = \lim_{k \rightarrow +\infty} -y_{m_k}. \\ \text{Case 4. } F'(0; h) &= \lim_{k \rightarrow +\infty} \frac{F(t_k h)}{t_k} = \lim_{k \rightarrow +\infty} -\frac{1}{q} y_{m_k} \quad \text{for some } q < z, \end{aligned}$$

for the respective sequences $t_k \downarrow 0$ and $\{y_{m_k}\}$.

On the other hand, by Proposition 2.4, the subsequences $\{y_{m_k}\}$ appearing in the above formulas are not weakly convergent.

Hence, we constructed a cone K and a K -convex mapping F on A which is not directionally differentiable at 0 in the direction h , which completes the proof. \square

6. Extension of mapping F . When X is a Hilbert space we can extend the mapping F from Theorem 5.2 to the half-space $H := \{x \in X : \langle x, h \rangle \geq 0\}$, where h is as in Theorem 5.2.

PROPOSITION 6.1. *Let X be a Hilbert space and $h \in X \setminus \{0\}$. Let set A , mapping $F : A \rightarrow Y$, and cone K be defined as in the proof of Theorem 5.2. Then there exists a convex extension $\bar{F} : H \rightarrow Y$ of F to the half-space $H = \{x \in X : \langle x, h \rangle \geq 0\}$.*

Proof. Let $h \neq 0$, set A , cone K , and mapping F be defined as in the proof of Theorem 5.2. Let $\bar{F} : H \rightarrow Y$ be defined as follows

$$\bar{F}(x) = F(\langle x, h \rangle h).$$

Let $x_1, x_2 \in H$ and $\lambda \in [0, 1]$. K -convexity of mapping \bar{F} on H follows from K -convexity of mapping F on A ,

$$\begin{aligned} \bar{F}(\lambda x_1 + (1 - \lambda)x_2) &= F(\langle \lambda x_1 + (1 - \lambda)x_2, h \rangle h) = F(\lambda \langle x_1, h \rangle h + (1 - \lambda) \langle x_2, h \rangle h) \\ &\leq_K \lambda F(\langle x_1, h \rangle h) + (1 - \lambda) F(\langle x_2, h \rangle h) = \lambda \bar{F}(x_1) + (1 - \lambda) \bar{F}(x_2). \end{aligned}$$

\square

In view of the above Proposition we can formulate the following theorem.

THEOREM 6.2. *Let X be a Hilbert space and let Y be a Banach space. If for every closed convex cone $K \subset Y$, and every K -convex mapping $F : H \rightarrow Y$ on $H = \{x \in X : \langle x, h \rangle \geq 0\}$ for all $h \in X$, $\|h\| = 1$ the directional derivative $F'(0; h)$ exists, then Y is weakly sequentially complete.*

Proof. Without loss of generality, in Theorem 5.2, we can define K -convex mapping F on $A = \{x \in X : x = rh, r \geq 0\}$, where direction $h \in X$ is such that $\|h\| = 1$. From Proposition 6.1 we can extend the K -convex mapping F on A to the K -convex mapping \bar{F} on H . Since, $\bar{F}(0) = 0$ the directional derivative of \bar{F} at 0 in the direction h is equal to

$$\lim_{t \downarrow 0} \frac{\bar{F}(\langle th, h \rangle h)}{t} = \lim_{t \downarrow 0} \frac{\bar{F}(t\|h\|^2 h)}{t} = \lim_{t \downarrow 0} \frac{F(th)}{t}.$$

The directional derivative of F at zero in the direction $h \in X$ does not exist which completes the proof. \square

7. Comments. In [23] directional derivatives of cone convex mappings are defined via the concept of infimum of the set $\left\{ \frac{F(x_0 + th) - F(x_0)}{t} : t > 0 \right\}$. Let K be a pointed cone. An element a is the infimum of A if a is a lower bound, i.e. $a \leq_K x \forall x \in A$ and, for any lower bound a' of A , we have $a' \leq_K a$ (c.f. [4] Ch.7, par.1, nr 8, [15] Ch.2, p.18). Without additional assumptions about cone K one cannot expect the equality

$$(7.1) \quad \lim_{t \downarrow 0} \frac{F(x_0 + th) - F(x_0)}{t} = \inf \left\{ \frac{F(x_0 + th) - F(x_0)}{t} : t > 0 \right\}.$$

In [15] we can find the following proposition.

PROPOSITION 7.1. *Let (X, τ) be a Hausdorff topological vector space ordered by the closed convex pointed cone K . If the net $(x_i)_{i \in I} \subset X$ is nonincreasing and convergent to $x \in X$, then $\{x_i : i \in I\}$ is bounded below and $x = \inf\{x_i : i \in I\}$.*

As a corollary we get

PROPOSITION 7.2. *If there exists $F'(x_0; h)$, there exists*

$$\inf \left\{ \frac{F(x_0 + th) - F(x_0)}{t} : t > 0 \right\}.$$

On the other hand, in [21] we can find the following example. Let $X = l^\infty$ and let us consider partial order generated by the cone $l_+^\infty := \{x \in l^\infty : x^k \geq 0 \forall k \geq 1\}$, l_+^∞ is a pointed closed convex cone. The sequence $\{x_n\} \subset l^\infty$ defined (for n fixed)

$$x_n^k := \begin{cases} -1 & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n, \end{cases} \text{ is nonincreasing, and } \inf\{x_n : n \geq 1\} = (-1, -1, -1, \dots).$$

But $\{x_n\}$ does not converge to its infimum.

We prove the cone convexity of the mapping F constructed in the proof of Theorem 5.2 (and in the proof of Theorem 6.2) for the cone K

$$K = \{y \in Y \mid z^*(y) \geq 0 \text{ for all } z^* \in K^*\}.$$

This cone is not pointed, so the concept of the infimum of the set is not defined. Let us note that the concept of directional derivative introduced in Definition 5.1 is well defined independently on whether cone K is pointed or not.

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